

A General Relativistic Model for Confinement in $SU(2)$ Yang-Mills Theory

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Abstract

In this paper we present a model of confinement based on an analogy with the confinement mechanism of the Schwarzschild solution of general relativity. Using recently discovered exact, Schwarzschild-like solutions of the $SU(2)$ Yang-Mills-Higgs equations we study the behaviour of a scalar, $SU(2)$ charged test particle placed in the gauge fields of this solution. We find that this test particle is indeed confined inside the color event horizon of our solution. Additionally it is found that this system is a composite fermion even though there are no fundamental fermions in the original Lagrangian.

I. INTRODUCTION

Recently a new classical solution was discovered by one of the authors [1] for an SU(2) Yang-Mills-Higgs system, which could be considered the Yang-Mills version of the Schwarzschild solution of general relativity. By using the Wu-Yang ansatz [2]

$$\begin{aligned} W_i^a &= \epsilon_{aij} \frac{r^j}{gr^2} [1 - K(r)] \\ W_0^a &= \frac{r^a}{gr^2} J(r) \\ \phi^a &= \frac{r^a}{gr^2} H(r) \end{aligned} \tag{1}$$

in the Euler-Lagrange equations for an SU(2) gauge field (*i.e.* W_μ^a) coupled to a triplet scalar field (*i.e.* ϕ^a) in the BPS limit [3] [4], a solution of the following form was found

$$\begin{aligned} K(r) &= \frac{Cr}{1 - Cr} \\ J(r) &= \frac{B}{1 - Cr} \\ H(r) &= \frac{A}{1 - Cr} \end{aligned} \tag{2}$$

Similiar solutions to pure Yang-Mills theory were discovered separately by Lunev [5], Mahajan and Valanju [6], and Swank *et. al.* [7]. The constants A and B must satisfy $A^2 - B^2 = 1$, and the constant C sets the size scale of the solution. The gauge and scalar fields of this solution develop singularities at $r = 0$ and also on a spherical shell $r = r_0 = 1/C$. This is also what happens with the Schwarzschild solution of general relativity in Schwarzschild coordinates. (For the Schwarzschild solution however the singularity on the spherical shell is an artifact of the coordinates which are used, as can be seen by using Kruskal coordinates. The singularities of our solution are true singularities, and can be thought of as the locations of the color charge of the system, in the same way that the singularity of the Coulomb potential of electromagnetism is the location of electric charge). The similiarity between the Schwarzschild solution and our solution can further be seen by comparing the connection coefficients of the Schwarzschild solution with the gauge fields from Eqs. (1) (2).

What is the physical significance of these new solutions ? Aside from indicating that there may be some deeper connection between general relativity and non-Abelian gauge theories [9] [10], it was speculated that these field configurations may give a confinement mechanism for Yang-Mills theories in general, and QCD in particular. There were several arguments given in Ref. [1] as to why one gets confinement from these solutions. First, in analogy with the Schwarzschild solution, which permanently traps any particle carrying gravitational charge (*i.e.* mass-energy) that crosses the event horizon, our solutions would classically trap any test particle with color charge which went inside $r_0 = 1/C$ (since the SU(2) gauge theory we are considering is a vector theory one can have repulsion as well as attraction, so there would be cases where a test particle would be permanently excluded from the region $r \leq 1/C$). Second, our solution has a structure similar to phenomenological bag models which are used to investigate hadron dynamics. (For a review of bag models see Ref. [11]). Finally, Ref. [12] reviews an argument for color confinement where the QCD vacuum is treated as analogous to a near perfect dia-electric medium, with $\epsilon_{medium} \ll 1$. In this scenario a spherical hole is postulated to exist inside the dia-electric vacuum, and a charge is placed at the center of the hole. This central charge will induce a charge on the surface of the spherical hole. Since the dielectric constant is taken as less than unity the induced charge has the same sign as the central charge, which makes this configuration stable against collapse (unlike the usual case where $\epsilon_{medium} > 1$ so that the induced charge and the central charge have opposite signs). This is similar to the configuration of our solution, if one takes the singularities of our solution to indicate the locations of color charge.

In this paper we would like to examine the question of whether this solution displays confinement in some more detail. In analogy with the hydrogen atom bound state system, we will treat the solution of Eqs. (1) (2) as a spinless “particle” which produces the Schwarzschild-like Yang-Mills field configuration around it. This is the interpretation given for a similar singular solution in Ref. [7]. In this way the “particle” is taken to be located at the field singularities (in the same way that the proton is taken to be located at the singularity of the Coulomb potential) while the gauge fields are taken as a background po-

tential whose energy is not included in the problem. The reason for taking this approach is that the field energy of the Schwarzschild-like solution is infinite because of the singularities. In the case of other known solutions of the SU(2) Yang-Mills-Higgs system (*e.g.* the 't Hooft-Polyakov monopole [13] and the BPS dyon [3] [4]) this problem does not arise, since these solutions have finite energy, and thus can be taken as particles in a straight forward way. However, it can be shown [7] that it is impossible for these finite energy solutions to form bound states with a test charge. For the Schwarzschild-like solutions we will find that not only is the test particle bound by the background potential field, but it is permanently confined inside the sphere $r = 1/C$. Throughout this paper we will ignore any quantum corrections to the classical solutions. Swank *et. al.* [7] have pointed out that $q\bar{q}$ creation near $r = 1/C$ would tend to decrease the strength of the barrier presented by the singularity. Mahajan and Valanju [6] [8] smooth the singularity on the sphere by giving a phenomenological parameterization of the quantum effects, and claim that in many cases there is still a substantial barrier at $r = 1/C$. Without a full quantum treatment of these solutions it is not possible to give any definite answer of how much the quantum effects will alter the nature of the classical solutions. There are some known methods [14] for quantizing the finite energy monopole and dyon solutions, which may be applicable to the Schwarzschild-like solutions. For simplicity we will take time component of the gauge field, W_0^a , to be zero by choosing $B = 0$ so that $A = 1$, and we will take our test particle to be spin-0. Then in order to obtain the motion of the test particle in the gauge potentials, we study the Klein-Gordon equation of the particle, minimally coupled to the fields of Eqs. (1), (2). The resulting equation can be reduced to a one dimensional Schrödinger-like equation. Solving this equation shows that the test particle does indeed remain trapped inside the region $r \leq 1/C$. In the process of reducing the minimally coupled Klein-Gordon equation to a one dimensional Schrödinger-like equation we will find that the total angular momentum of our system equals the usual orbital angular momentum plus the isospin of the test particle. Therefore this system can have a spin 1/2 even though it contains only bosons. This effect is just the spin from isospin mechanism discussed by Jackiw, Rebbi, 't Hooft and Hasenfrantz [15]. In Refs. [15] it was shown

that bringing a particle with isospin 1/2 into the presence of a 't Hooft-Polyakov monopole resulted in the total system having the spin of a fermion. That the same thing happens in our case should not be a surprise, since our solution has the same mixing of spatial and group indices – see Eq. (1) – that resulted in the spin 1/2 in Refs. [15]. One advantage of our composite system over those considered in Refs. [15] is that the Schwarzschild-like field configuration provides its own binding mechanism. In the case of an isospinor particle moving in the field of a 't Hooft-Polyakov monopole one has to postulate some additional, phenomenological binding force to bind the isospinor and monopole together. It has also been shown [16] that such composite spin 1/2 systems obey Fermi-Dirac statistics. Thus the bound state system that we consider in this paper, consisting of a scalar particle with color charge moving in the potential of the Schwarzschild-like solution, is actually a composite fermion even though the original Lagrangian contains only bosonic fields.

II. QUANTUM MOTION OF SCALAR, SU(2) CHARGED TEST PARTICLE

The motion of a scalar particle carrying SU(2) charge, in presence of gauge fields W_i^a (where $W_0^a = 0$) is given by the minimally coupled Klein-Gordon equation

$$\left(\partial_i - \frac{i}{2}g\sigma^a W_i^a\right)\left(\partial_i - \frac{i}{2}g\sigma^a W_i^a\right)_B^A \Phi^B(x) = -(E^2 - m^2)\Phi^A(x) \quad (3)$$

where the scalar field was taken to have the usual time dependence - $\Phi^A(x, t) = e^{-iEt}\Phi^A(x)$. The scalar particle $\Phi^A(x)$ is in the fundamental representation of SU(2) and has a mass m . The matrices $(\sigma^a)_B^A$ are the Pauli matrices with $a = 1, 2, 3$. and $A, B = 1, 2$. E is the total energy of the scalar particle. Substituting the Wu-Yang ansatz for the gauge fields into this gives

$$\left(\nabla^2 - \frac{[1 - K(r)]}{r^2}\sigma^a l^a - \frac{[1 - K(r)]^2}{2r^2}\right)_B^A \Phi^B(x) = -(E^2 - m^2)\Phi^A(x) \quad (4)$$

where $l_a = -i\epsilon_{aij}r^i\partial^j$ is the orbital angular momentum operator. In order to deal with the $\sigma^a l^a$ term we define the following operator

$$J^a = l^a + \frac{1}{2}\sigma^a \quad (5)$$

This operator, which is a combination of the orbital angular momentum operator and the isospin operator of the particle $\Phi^A(x)$, is in fact the total angular momentum of the system, since it commutes with the Hamiltonian of the system. In addition J^a obeys the usual commutation relationships of total angular momentum – $[J^a, J^b] = i\epsilon^{abc} J^c$ and $[J^a, P^b] = i\epsilon^{abc} P^c$, where P^a is the canonical momentum. A more thorough demonstration that J^a is indeed the total angular momentum of the system is given in Ref. [15].

Eq. (4) can be reduced to a one dimensional Schrödinger-like equation by taking $\Phi^A = \frac{1}{r}f_{Jl}(r)Y_{JlM}^A(\theta, \phi)$, where $f_{Jl}(r)$ is a radial function, and $Y_{JlM}^A(\theta, \phi)$ are the spherical harmonics associated with the operator J^a of Eq. (5). These spherical harmonics obey the usual operator eigenvalue equations – $J_{op}^2 Y_{JlM}^A = J(J+1)Y_{JlM}^A$ and $l_{op}^2 Y_{JlM}^A = l(l+1)Y_{JlM}^A$. Using all this in Eq. (4) yields

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{J(J+1) - l(l+1) - \frac{3}{4}}{r^2} [1 - K(r)] - \frac{[1 - K(r)]^2}{2r^2} \right) f_{Jl}(r) = - (E^2 - m^2) f_{Jl}(r) \quad (6)$$

Using $K(r) = Cr/(1 - Cr)$, making a change of variables to $x = Cr$ and collecting terms yields the final form of the Schrödinger-like equation which we wish to solve.

$$\left(\frac{d^2}{dx^2} - \frac{Ex^2 + Fx + G}{x^2(1-x)^2} \right) f_{Jl}(x) = -\frac{D}{C^2} f_{Jl}(x) \quad (7)$$

where

$$\begin{aligned} D &= E^2 - m^2 \\ E &= -2J(J+1) + 3l(l+1) + 7/2 \\ F &= 3J(J+1) - 5l(l+1) - 17/4 \\ G &= -J(J+1) + 2l(l+1) + 5/4 \end{aligned} \quad (8)$$

Eq. (7) is a simple second order differential equation, which looks like a one dimensional Schrödinger equation with the first term on the left hand side as the kinetic term, and the

second term on the left as the potential term. The term on the right hand side acts as the energy eigenvalue. A typical example of the form of the “potential” is given in Fig. (1), which shows the $l = 0, J = 1/2$ and $l = 1, J = 1/2$ cases. From this, one can see that the particle will remain trapped between the two barriers at $x = 0$ and $x = 1$. This problem is similar to the Pöschl-Teller potential hole [17], which also confines a particle between two singularities that are of the same order as those in our problem. Unfortunately, when one attempts to find a power series solution for Eq. (7) an extremely complicated recursion relationship occurs which makes it difficult to get much physical insight from the analytic solution. This makes it more convenient (without sacrificing any physical insight) to obtain the solutions numerically. Figs. (2) and (3) show the radial functions, $f_{Jl}(x)$, of the first two energy levels for $J = 1/2, l = 0$ and $J = 1/2, l = 1$ respectively. From these figures one can see that the scalar particle is confined between the two singularities at $x = 0$ and $x = 1$. The radial function f_{Jl} (and therefore Φ^A) becomes zero for $r > r_0$. Since we solved the one dimensional Schrödinger equation numerically we needed some method for taking the singularity at $x = 1$ into account. This was accomplished by requiring that $f_{Jl}(x)$ be zero if one of our mesh points was on the singularity. This condition is exactly the same as in ordinary quantum mechanics where the wavefunction is required to vanish where ever the potential becomes infinite. Pinning f_{Jl} to be zero at $x = 1$ and solving Eq. (7) for the range $0 \leq x \leq 2$ results in the particle being entirely contained in the region $x < 1$.

In Table (I) we list the first four energy levels for the “potentials” that results by setting $l = 0, 1, 2, 3$. The eigenvalues for the $l = 0, J = 1/2$ state look similar to the energy spectrum for the one dimensional nonrelativistic infinite square well (*i.e* the first excited state is approximately 4 times the ground state, the second excited state is approximately 9 times the ground state, *etc.*) except that the eigenvalues are the square of the energy, E^2 , rather than the energy, E . This is not surprising since the $l = 0, J = 1/2$ “potential” in Fig. (1) is a reasonable approximation of a square well, and Eq. (7) is similar to the one-dimensional Schrödinger equation. The spectrum of eigenvalues given in Table (I) agrees roughly with those obtained by Mahajan and Valanju [6] who studied the motion of fermions

in a similar, SU(3) background field.

Since the composite system of the particle Φ^A moving in the fields of the Schwarzschild-like solution behaves as a fermion one could try to construct a model for baryons from this system. These toy model SU(2) baryons are bound states of two scalar particles (the test particle, Φ^A , and the Schwarzschild-like solution considered as a “particle”) whose internal angular momentum comes from the spin from isospin mechanism. This is to be compared to the usual picture of baryons as bound states of three fundamental fermions (*i.e.* quarks). In either case one can not separate the constituents of the bound state. Although a more realistic model should use SU(3) as the gauge group rather than SU(2), the present development should give a good qualitative idea of the structure of these scalar-scalar bound state baryons.

First, assuming that most of the mass of the real baryons comes from binding energy, we will take the mass of the test particle, Φ^A , to be small so that $D = E^2 - m^2 \approx E^2$. In order to get a numerical value for the energy eigenvalues from Table (I), we must choose a value for C , which is equivalent to choosing a radius for spherical shell singularity of the gauge fields. Taking this radius to be 1 fermi leads to $C \approx 200$ MeV in our units. For the first four states of the $l = 0$, $J = 1/2$ system we find $E_0 = 716$ MeV, $E_1 = 1394$ MeV, $E_2 = 2044$ MeV, and $E_3 = 2685$ MeV. Similarly for the first four states of the $l = 1$, $J = 1/2$ system we find $E_0 = 920$ MeV, $E_1 = 1661$ MeV, $E_2 = 2341$ MeV, and $E_3 = 3000$ MeV. Since the energy scales with C as $E = C\sqrt{N}$ (where N is the numerical value of the eigenvalue, $D/C^2 \approx E^2/C^2$, given in Table (I)) one could increase (decrease) these energies by decreasing (increasing) the radius of the spherical singularity. In order to calculate the mass of this composite system it would be necessary to add the constituent mass of the Schwarzschild “particle” to the binding energy. As mentioned in the introduction, we are treating the gauge fields of Eqs. (1) (2) as background potentials, since including their energy in the mass of the Schwarzschild “particle” would lead to an infinite mass. What one can do is to take the constituent mass of the Schwarzschild “particle” as a parameter, which is fixed by the measured mass of the lowest mass state. In this way one can not calculate

the mass of the lowest state from first principles, but the masses of all the other excited states can be calculated. In a more detailed scheme one could also take C and the mass of the test particle, m , as free parameters which are fixed using the first few states as inputs. This may seem a somewhat shady procedure, but it is not too different from what is done for other bound state systems such as the hydrogen atom. In the case of the hydrogen atom one calculates the motion of the electron in the Coulomb potential produced by the proton. The mass of the hydrogen atom is then found by adding the mass of the proton, the mass of the electron and the binding energy, in the approximation of taking the proton to be a point charge. The energy of the Coulomb field due to the proton (which would give an infinite contribution to the mass of the system if treated classically) is in effect normalized into the mass of the proton. It is worth noting that if one wanted to identify the 716 MeV energy state of the above baryons with the proton most of the mass would be coming from the binding energy. The mass parameter of the Schwarzschild “particle” would then be ≈ 222 MeV. This justifies *a posteriori* having the nearly massless test particle, Φ^A , move in the stationary field of the Schwarzschild “particle”. This is in qualitative agreement with the quark model picture of the proton or neutron, where the quarks are given a small current mass, and most of the mass is attributed to the QCD binding.

Although this toy model of the baryons has some interesting features (a general relativistic explanation of confinement, and having the spin come from the isospin of the test particle) there are many problems and questions which must first be addressed before one could make a comparison with real baryons. First one should use SU(3) rather than SU(2) as the gauge group. We have recently discovered the SU(N) generalization for the Schwarzschild-like solution [18], so it should be possible to carry through the development here for SU(3). The SU(N) solution is simply an embedding of SU(2) into SU(N), so for SU(3) we would expect results which, even numerically, are not too different from those obtained in this paper. Comparing our eigenvalues from Table (I) with corresponding ones calculated by Mahajan and Valanju [6], for fermions moving in a similar SU(3) potential, we find that these eigenvalues are not drastically different. Second, these bound states do

not carry any electric charge, but do carry a topological magnetic charge due to the scalar field ϕ^a . The lack of an electric charge makes these states bad models for charged baryons such as the proton, while the presence of the magnetic charge makes them a bad model for any baryons. In order to give these bound states electric charge one could easily give the test particle an Abelian electric charge in addition to its SU(2) color charge. This added electric charge on the test particle should not change the nature of the composite system much, since the electric coupling is a small perturbation to the color SU(2) coupling. This still leaves the topological magnetic charge coming from the scalar field ϕ^a . There is a possible resolution to this. Instead of adding an Abelian electric charge by hand one could work with a dual non-Abelian theory, and then, according to the conjecture of Montonen and Olive [19], the topological charge of the solution becomes electric rather than magnetic. In this way the field configuration of our solution carries the electric charge, so there is no need to have the test particle carry the electric charge. Finally, there is the already mentioned problem of the mass of the Schwarzschild “particle”. If we equate the mass with the volume integral of the energy density we get an infinite mass from the singularities in the solution. Another possible resolution, aside from normalizing this energy into the constituent mass of the Schwarzschild “particle”, is to try and smooth out the singularities of the solution, while still maintaining the spherical barrier feature that leads to confinement. This can be done by hand by allowing $1/C$ to be complex [6]. A less *ad hoc* approach would be to allow the scalar field ϕ^a to have a mass and/or self coupling term in the hope that the solution with these terms present would be smooth. This is what happens in the case of the BPS monopole as compared to the ’t Hooft-Polyakov monopole – by letting the scalar field have a mass and a self coupling the singularity at $r = 0$ of the BPS monopole gets smoothed out in the ’t Hooft-Polyakov monopole. However when the scalar field is allowed to have a mass and self coupling term one must look for solutions numerically.

III. DISCUSSION AND CONCLUSIONS

Using recently discovered solutions to the SU(2) Yang-Mills-Higgs system we studied the behaviour of a scalar, SU(2) charged test particle in the background potential of these solutions. The main goal of this paper was to show that these Schwarzschild-like solutions did exhibit confinement in that they kept the test particle restricted to the region $r \leq 1/C$. By minimally coupling the Klein-Gordon equation of the test particle, Φ^A , to the gauge fields of the solution, we obtained a Schrödinger-like equation whose potential term had two infinite barriers - one at $r = 0$ and the other at $r = 1/C$. These barriers confined the test particle to remain in the region $0 < r < 1/C$. In addition to confining the test particle, the field configuration of the Schwarzschild-like solution converted the isospin of the test particle into real spin. This spin from isospin effect is the same that occurs when one places a scalar SU(2) test particle in the field configuration of a 't Hooft-Polyakov monopole. (This is further related to the old result in electromagnetism that the fields of an electric charge and a magnetic monopole carry angular momentum [20]). Goldhaber [16] has shown that such bound states, in addition to carrying the angular momentum associated with fermions, also obey Fermi-Dirac statistics. Therefore our bound state system, consisting of the spinless test particle moving inside the field configuration of Eqs. (1) and (2), is a composite fermion, which results from a theory that originally contained no spin-1/2 fields. To construct composite, integer spin particles in the present model, one should place spin 1/2, test particles inside the fields of the Schwarzschild-like potential. The advantage these present composite fermions (and bosons) have over those discussed in Refs. [15] is that they provide for their own binding mechanism. For the 't Hooft-Polyakov monopole and the BPS dyon it has been demonstrated [7] that one can not form bound states with these field configurations. In order to get a test particle to form a bound state with these finite energy solutions one needs to postulate some phenomenological Yukawa binding between the two scalar particles. In the present case the isospin 1/2, scalar particle is automatically bound by the field configuration of our solution. Not only is it bound, but it is permanently confined.

Thus, even though the system is a composite of the scalar test particle and the “particle” represented by our solution, one can never separate the system into its constituent parts.

There are several possible extensions to this work. One could study the behaviour of color charged fermions in the background field of our solution. These states would be bosons by the same mechanism that made the scalar particles of this paper fermions. Part of the reason for studying scalar particles here was because of this spin from isospin effect. (If we had worked with colored fermions the spin from isospin effect would still have been present, however the composite system would then have had integer spin due to the fundamental spin 1/2 of the fermion. It is much more unusual to have a spin 1/2 composite system built from scalars, rather than an integer spin composite system built from fermions). Another possibility which was not explored in this paper is to have a phenomenological Yukawa coupling between the scalar, isovector field, ϕ^a and the scalar, isospinor field, Φ^A . We felt that it was one of the strong points of our Schwarzschild-like solution that it did not require such a coupling in order to bind the isospinor particle to the Schwarzschild-like field configuration. In this paper we looked at a special case of the Schwarzschild-like solution, namely the case where $W_0^a = 0$. It may be worthwhile to examine the more general case, $\phi^a \neq 0$ and $W_0^a \neq 0$. Finally our Schwarzschild-like solutions were found in the BPS limit of the field ϕ^a . It may be of interest to see if one can find, even numerically, solutions with a nonzero mass and/or self-interaction term for the scalar triplet, ϕ^a , which have the confining sphere feature of Eqs. (1) , (2). The hope is that allowing for a nonzero mass and/or self-interaction may smooth out some or all of the singularities of the solution in the same way that the singularity in the magnetic field, at $r = 0$, of the BPS dyon gets smoothed over in the 't Hooft-Polyakov monopole. If this conjecture proves to be true then this might provide a classical fix for the problems posed by the singularities of the solution.

Our solution is completely classical. One should study the quantum fluctuations around this classical solution. There are known methods for doing this [14], and such an investigation is currently underway.

IV. ACKNOWLEDGEMENTS

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Figure Captions

Figure 1 : The “potentials” of the one dimensional Schrödinger equation, Eq. (7), for the specific cases $l = 0, J = 1/2$ and $l = 1, J = 1/2$.

Figure 2 : The radial functions, $f_{Jl}(x)$, for the three lowest states for the case $l = 0, J = 1/2$.

Figure 3 : The radial functions, $f_{Jl}(x)$, for the three lowest states for the case $l = 1, J = 1/2$.

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TABLES

TABLE I. This table gives the eigenvalues, D_i/C^2 , to Eq. (7), for $l = 0, 1, 2, 3$.

l	J	D_0/C^2	D_1/C^2	D_2/C^2	D_3/C^2
0	1/2	12.81	48.57	104.47	180.23
1	1/2	21.16	68.97	136.97	225.02
1	3/2	23.15	64.48	125.52	206.31
2	3/2	31.18	88.65	165.89	263.08
2	5/2	40.06	90.57	160.58	250.25
3	5/2	43.37	111.22	198.15	304.86
3	7/2	60.87	121.11	200.88	300.18